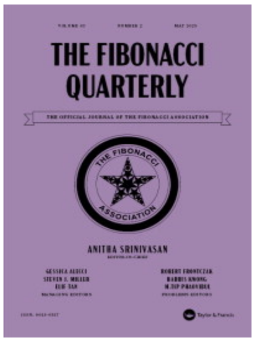

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PERRIN NUMBERS WHICH ARE THE SUM, DIFFERENCE, OR PRODUCT OF TWO FIBONACCI NUMBERS

BOUAZZAOUI ZAKARIAE, BOUGHADI ZOUHAIR, AND EL HABIBI ABDELAZIZ

ABSTRACT. Let $(E_n)_{n\geq 0}$ be the Perrin sequence given by $E_{n+3} = E_{n+1} + E_n$, with the initial conditions $E_0 = 3, E_1 = 0$ and $E_2 = 2$. The aim of this paper is to find all Perrin numbers which are the sum, difference or product of two Fibonacci numbers by using the methods of Baker-Davenport. We prove the finitude of the number of solutions, which we describe explicitly in each equation.

KEYWORDS. Fibonacci numbers, Perrin numbers, Linear form in logarithms, Reduction method.

1. INTRODUCTION

In the past few decades, questions of solvability of Diophantine equations have seen considerable advances, especially after the affirmative answer to Fermat's last theorem. The proof given by Wiles [16] uses tools from several fields of mathematics, including representation theory and arithmetic geometry, as well as the modular approach of Wiles works for large classes of equations arising from elliptic curves or related to modularity results of elliptic curves. As an example, for Fibonacci sequence $(F_n)_{n\geq 0}$ and Lucas sequence $(L_n)_{n\geq 0}$, the equations like $F_n = y^p$ and $L_n = y^p$ are dealt with in [7], [4] and subsequent papers, where modularity results are used, and Baker's theory of linear forms in logarithms played an important role.

The methods of Baker-Davenport ([5]) give bounds for exponents, which are usually very large, which can be reduced with the help of specific computational algorithms. These are highly effective in the class of Diophantine equations arising from recurrence sequences, where the methods prove the finitude of the number of solutions, like the example studied in this paper. Here we focus on Perrin and Fibonacci sequences.

Recall that $(E_n)_{n\geq 0}$ is the Perrin sequence given by

$$E_{n+3} = E_{n+1} + E_n,$$

with the initial conditions $E_0 = 3, E_1 = 0$ and $E_2 = 2$. Although the sequence is named after R. Perrin who studied it in 1899 ([10]), it had been explored earlier, in 1876, by Edouard Lucas. The following is the list of a few Perrin numbers:

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209,$$
$$277, 367, 486, 644, 853, \cdots$$

It is the sequence A001608 in the OEIS ([15]). For a subsequent paper on Perrin numbers and their properties see [14].

In [13], the authors found all solutions of the Diophantine equation $E_s = F_n^{(k)}$, where $F_n^{(k)}$ is the k-generalized Fibonacci sequence. This paper is an addition to the growing literature around the study of Diophantine properties of certain linear recurrence sequences. More specifically, we are interested in the following two diophantine equations

$$E_s = F_n \pm F_m, \tag{1}$$

and

$$E_s = F_n F_m, \tag{2}$$

where n, m, and s are nonnegative integers. Precisely, we prove the following three theorems.

Theorem 1.1. *The Perrin numbers which are the sum of two Fibonacci numbers are:* $0, 2, 3, 5, 7, 10, 22, 29, 39, 68, 90, 644$. *Namely, we have:*

$$\begin{aligned}
0 =& E_1 = 2F_0,\\
2 =& E_2 = E_4 = 2F_1 = F_2 + F_1 = 2F_2 = F_3 + F_0,\\
3 =& E_0 = E_3 = F_3 + F_1 = F_3 + F_2 = F_4 + F_0,\\
5 =& E_5 = E_6 = F_4 + F_3 = F_5 + F_0,\\
7 =& E_7 = F_5 + F_3,\\
10 =& E_8 = 2F_5 = F_6 + F_3,\\
22 =& E_{11} = F_8 + F_1 = F_8 + F_2,\\
29 =& E_{12} = F_8 + F_6,\\
39 =& E_{13} = F_9 + F_5,\\
68 =& E_{15} = 2F_9 = F_{10} + F_7,\\
90 =& E_{16} = F_{11} + F_1 = F_{11} + F_2,\\
644 =& E_{23} = F_{15} + F_9.
\end{aligned}$$

Theorem 1.2. *The Perrin numbers which are the difference of two Fibonacci numbers are:* $0, 2, 3, 5, 7, 10, 12, 29, 68$. *Namely, we have:*

$$\begin{aligned}
0 =& E_1 = F_2 - F_1,\\
2 =& E_2 = E_4 = F_3 - F_0 = F_4 - F_1 = F_4 - F_2 = F_5 - F_4,\\
3 =& E_0 = E_3 = F_4 - F_0 = F_5 - F_3 = F_6 - F_5,\\
5 =& E_5 = E_6 = F_5 - F_0 = F_6 - F_4 = F_7 - F_6,\\
7 =& E_7 = F_6 - F_1 = F_6 - F_2,\\
10 =& E_8 = F_7 - F_4,\\
12 =& E_9 = F_7 - F_1 = F_7 - F_2,\\
29 =& E_{12} = F_9 - F_5,\\
68 =& E_{15} = F_{11} - F_8.
\end{aligned}$$

Theorem 1.3. *The Perrin numbers which are the product of two Fibonacci numbers are:* $2, 3, 5, 10, 39, 68$. *Namely, we have the following:*

$$\begin{aligned} 2 =& E_2 = E_4 = F_3 \times F_1 = F_3 \times F_2, \\ 3 =& E_0 = E_3 = F_4 \times F_1 = F_4 \times F_2, \\ 5 =& E_5 = E_6 = F_5 \times F_1 = F_5 \times F_2, \\ 10 =& E_8 = F_5 \times F_3, \\ 39 =& E_{13} = F_7 \times F_4, \\ 68 =& E_{15} = F_9 \times F_3. \end{aligned}$$

The same methods apply to find all integer solutions (s, m, n, r) of equations like

$$E_s = F_n \pm F_m \pm F_r, \tag{3}$$

and

$$E_s = F_n F_m F_r, \tag{4}$$

see for instance [1], [11], [2] and subsequent references. Similar equations are previously considered by other authors (e.g., [12], [6], [8] and subsequent papers). For computational purposes, we choose to consider in this paper the case of equations (1) and (2).

2. Auxiliary results

2.1. **Linear forms in logarithms and the Baker-Davenport reduction method.** The proofs of our main theorems use lower bounds for linear forms in logarithms of algebraic numbers and a version of Baker-Davenport reduction method. Let us recall the results used throughout this paper.

For any nonzero algebraic number δ of degree d over the field of rational numbers $\mathbb{Q}$, let $a \prod_{i=1}^{d} \left(X - \delta^{(i)}\right)$ be the minimal polynomial of δ over $\mathbb{Z}$ (with $a > 0$), we denote by

$$h(\delta) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log \max\left(1, |\delta^{(i)}|\right)\right)$$

the usual absolute logarithmic height of δ. The following properties of logarithmic height are found in many works stated in the references:

$$\begin{aligned} h(\delta \pm \xi) &\leq h(\delta) + h(\xi) + \log(2), \\ h(\delta\xi^{\pm 1}) &\leq h(\delta) + h(\xi), \\ h(\delta^k) &= |k|\, h(\delta). \end{aligned}$$

With the above notations, E.M. Matveev proved the following theorem (see [9]).

Theorem 2.1. *Let* $\delta_1, \ldots, \delta_l$ *be real algebraic numbers and let* $b_1, \ldots, b_l$ *be nonzero integers. Let* D *be the degree of the number field* $\mathbb{Q}(\delta_1, \ldots, \delta_l)$ *over* $\mathbb{Q}$. *If* $\delta_1^{b_1} \cdots \delta_l^{b_l} - 1 \neq 0$. *Then*

$$|\delta_1^{b_1} \cdots \delta_l^{b_l} - 1| \geq \exp(-1.4 \times 30^{l+3} \times l^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_l),$$

where

$$A_j = \max\{Dh(\delta_j), |\log \delta_j|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, l$$

and

$$B \geq \max\{|b_1|, \ldots, |b_l|\}.$$

The next lemma was proved by Dujella and Petho [5, Lemma 5]. This is a variation of a result of Baker and Davenport [3]. For a real number x, $\| x \|$ denotes the distance from x to the nearest integer, that is $\| x \| = \min\{|x-n| : n \in \mathbb{Z}\}$.

Lemma 2.2. *Let M be a positive integer, and let p/q be a convergent of the continued fraction of the irrational number ψ such that $q > 6M$, and let $A > 0$, $B > 1$ and μ be real numbers. Let $\varepsilon := \| \mu q \| - M \| \psi q \|$. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\psi - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

We remark that Lemma 2.2 cannot be applied when $\varepsilon < 0$. For this case, we use the following well-known technical result from Diophantine approximation, known as Legendre's criterion.

Lemma 2.3. *(Legendre) Let κ be a real number and x, y integers such that*

$$\left|\kappa - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Then $\frac{x}{y} = \frac{p_\kappa}{q_\kappa}$ is a convergent of κ. Furthermore, let M and N be nonnegative integers such that $q_N > M$. Let $[a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of κ and put $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$. Then the inequality

$$\left|\kappa - \frac{x}{y}\right| \geq \frac{1}{(a(M)+2)y}$$

holds for all pairs (x, y) of positive integers with $0 < y < M$.

2.2. **Properties of Fibonacci and Perrin sequences.**

In this subsection we recall necessary facts about Fibonacci and Perrin numbers that will be used in the following.

Fibonacci numbers. The characteristic polynomial of the Fibonacci sequence is $x^2 - x - 1$ and its roots are denoted $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Thus, for any integer $n \geq 0$, the Binet formula for F_n gives

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

We can prove by induction that we have

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}. \tag{5}$$

We remark that

$$1 < \alpha < 2 \quad \text{and} \quad \frac{1}{2} < |\beta| < 1. \tag{6}$$

Perrin numbers. The Binet formula for the Perrin sequence is

$$E_n = \gamma^n + \eta^n + \rho^n, \tag{7}$$

where γ, η and $\rho = \overline{\eta}$ are the roots of the polynomial $x^3 - x - 1$. Precisely,

$$\gamma = \frac{r_1 + r_2}{6} \text{ and } \eta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

where $r_1 = \sqrt[3]{108 + 12\sqrt{69}}$ and $r_2 = \sqrt[3]{108 - 12\sqrt{69}}$.

We make the following useful observation,

$$\gamma^{n-2} \le E_n \le \gamma^{n+1}, \quad \text{for all } n \ge 2. \tag{8}$$

Furthermore, we have

$$1.32 < \gamma < 1.33 \quad \text{and} \quad 0.86 < |\eta| = |\rho| < 0.87. \tag{9}$$

3. Perrin numbers as the sum or difference of two Fibonacci numbers

In this section we prove Theorems 1.1 and 1.2. For this, we first show that solutions (n, m, s) of equation (1) satisfy $m \le n < 196$ and $s < 396$, then we use the computer to check such solutions.

Let us now consider equation (1). We start by comparing s and n. By (8) and (5) we obtain

$$\gamma^{s-2} < E_s = F_n \pm F_m \le F_n + F_m < 2\alpha^{n-1} < \alpha^{n+1},$$

which gives that

$$(s-2)\log\gamma < (n+1)\log\alpha.$$

Since $\dfrac{\log\alpha}{\log\gamma} < 2$, we get

$$s < 2n + 4. \tag{10}$$

3.1. **The initial bound on n.** In this section we give a bound for the value of n for equation (1).

Using the Binet formulas of the Fibonacci and Perrin numbers, we write (1) in the following form:

$$\alpha^n - \sqrt{5}\gamma^s = \sqrt{5}\eta^s + \sqrt{5}\rho^s + \beta^n \pm \sqrt{5}F_m.$$

Thus, we have

$$\begin{aligned} \left|1 - \gamma^s\alpha^{-n}\sqrt{5}\right| &< \frac{1}{\alpha^n}(|\beta|^n + \sqrt{5}\alpha^{m-1} + 2\sqrt{5}|\eta|^s) \\ &< \frac{1}{\alpha^{n-m}}\left(\frac{1}{\alpha^m} + \frac{\sqrt{5}}{\alpha} + \frac{2\sqrt{5}}{\alpha^m}\right) < \frac{5}{\alpha^{n-m}}. \end{aligned}$$

We set $\Lambda_1 := \gamma^s\alpha^{-n}\sqrt{5} - 1$. Then we have

$$|\Lambda_1| < \frac{5}{\alpha^{n-m}}. \tag{11}$$

We now write equation (1) in the following way:

$$\frac{\alpha^n}{\sqrt{5}} \pm \frac{\alpha^m}{\sqrt{5}} - \gamma^s = \eta^s + \rho^s + \frac{\beta^n}{\sqrt{5}} \pm \frac{\beta^m}{\sqrt{5}}.$$

Hence

$$|\alpha^n \pm \alpha^m - \gamma^s\sqrt{5}| \le 2|\eta|^s\sqrt{5} + |\beta|^n + |\beta|^m < 2\sqrt{5} + 2 < 7.$$

Multiplying by $\frac{1}{\alpha^n}(1 \pm \alpha^{m-n})^{-1}$ (with the assumption $n \neq m$ in the case of the difference equation) we get

$$|1 - \gamma^s \sqrt{5} \alpha^{-n}(1 \pm \alpha^{m-n})^{-1}| < \frac{7}{\alpha^n} \cdot \frac{1}{1 \pm \alpha^{m-n}}.$$

To bound $\frac{1}{1 \pm \alpha^{m-n}}$ note that $\frac{1}{1 \pm \alpha^{m-n}} \leq \frac{1}{1 - \alpha^{m-n}}$, and observe that $\frac{1}{1 - \alpha^{m-n}} \leq \frac{\alpha}{\alpha - 1} < 3$. Hence, $\frac{1}{1 \pm \alpha^{m-n}} < 3$. Thus, we obtain

$$|\Lambda_2| < \frac{21}{\alpha^n}, \tag{12}$$

where $\Lambda_2 := \gamma^s \sqrt{5} \alpha^{-n}(1 \pm \alpha^{m-n})^{-1} - 1$.

Lemma 3.1. *If* (n, m, s) *are nonnegative integer solutions of the Diophantine equation (1) with* $m \leq n$*, then* $n < 3 \times 10^{31}$.

Proof. We obtain this bound of n after using Matveev's theorem to give a lower bound for $|\Lambda_1|$ and $|\Lambda_2|$. In order to do this we need the necessary data. We consider the number field $\mathbb{L} = \mathbb{Q}(\sqrt{5}, \gamma)$ which is of degree $D = 6$ over $\mathbb{Q}$. For the linear form Λ_1 we denote

$$\delta_1 = \gamma, \quad \delta_2 = \alpha, \quad \delta_3 = \sqrt{5}, \quad b_1 = s, \quad b_2 = -n \quad \text{and} \quad b_3 = 1.$$

Notice that

$$h(\delta_1) = \frac{\log \gamma}{3}, \quad h(\delta_2) = \frac{\log \alpha}{2} \quad \text{and} \quad h(\delta_3) = \log(\sqrt{5}).$$

Hence, we choose $A_1 = 0.6$, $A_2 = 1.5$, $A_3 = 4.9$, and $B = 2n + 4$.
Let us prove that $\Lambda_1 \neq 0$. For this, let $\mathbb{K}$ be the normal closure of $\mathbb{L}$. We consider the Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ which satisfies $\sigma(\gamma) = \eta$ and $\sigma(\alpha) = \alpha$. Suppose that $\Lambda_1 = 0$, hence $\gamma^s \alpha^{-n} \sqrt{5} = 1$. By applying the automorphism σ to this equality we obtain

$$\eta^s \alpha^{-n} \sqrt{5} = 1.$$

Using (9), for $n, s \geq 1$, we find that

$$\left| \eta^s \alpha^{-n} \sqrt{5} \right| < 1, \ \forall n, s \geq 1,$$

which is a contradiction with the assumption $\Lambda_1 = 0$.

Then we have the following bound

$$\begin{aligned} |\Lambda_1| &\geq \exp(-1.4 \times 30^6 \times 3^{4.5} \times 6^2(1 + \log(6))(1 + \log(2n + 4)) \\ &\quad \times 4.9 \times 1.5 \times 0.6) \\ &> \exp(-6.35 \times 10^{13}(1 + \log(2n + 4))). \end{aligned}$$

After taking the logarithm and using (11) we find that

$$(n - m) \log \alpha < 6.4 \times 10^{13}(1 + \log(2n + 4)). \tag{13}$$

For $\Lambda_2 := \gamma^s \sqrt{5} \alpha^{-n}(1 \pm \alpha^{m-n})^{-1} - 1$, we consider the following data

$$\delta_1 = \gamma, \quad \delta_2 = \alpha, \quad \delta_3 = \sqrt{5}(1 \pm \alpha^{m-n})^{-1}, \quad b_1 = s, \quad b_2 = -n \quad \text{and} \quad b_3 = 1.$$

As above we take

$$D = 6, \quad \ell = 3, \quad B = 2n+4, \quad A_1 = 0.6, \quad A_2 = 1.5 \quad \text{and} \quad A_3 = 9+3(n-m)\log\alpha.$$

Let us have a look on the logarithmic height of δ_3. We have

$$\begin{aligned} h(\delta_3) &\leq \log\sqrt{5} + |m-n|\frac{\log(\alpha)}{2} + \log 2 \\ &= \log(2\sqrt{5}) + (n-m)\frac{\log(\alpha)}{2}. \end{aligned}$$

Since

$$\delta_3 = \frac{\sqrt{5}}{1 \pm \alpha^{m-n}} < 3\sqrt{5} \quad \text{and} \quad \delta_3^{-1} = \frac{1 \pm \alpha^{m-n}}{\sqrt{5}} < \frac{2}{\sqrt{5}} < 2,$$

we have $|\log(\delta_3)| < 2$. Then we can take

$$\max\{6h(\delta_3), |\log\delta_3|, 0.16\} < A_3 := 9 + 3(n-m)\log(\alpha).$$

Note that we have $\Lambda_2 \neq 0$ in a similar way as for Λ_1. Applying Matveev's theorem we obtain the following bound for Λ_2:

$$\begin{aligned} |\Lambda_2| &\geq \exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 6^2(1+\log(6))(1+\log(2n+4)) \times 1.5 \times 0.6 \times A_3\right) \\ &> \exp\left(-1.3 \times 10^{13}(1+\log(2n+4)) \times A_3\right). \end{aligned}$$

Applying the logarithm and using (12) we obtain

$$n\log\alpha < 1.31 \times 10^{13}(1+\log(2n+4))(9+3(n-m)\log\alpha).$$

Using (13) we obtain that

$$9 + 3(n-m)\log\alpha < 1.93 \times 10^{14}(1+\log(2n+4)).$$

Therefore,

$$n\log\alpha < 2.53 \times 10^{27}(1+\log(2n+4))^2.$$

Notice that $2n+4 \leq 6n$ for $n \geq 1$, then we have

$$n\log\alpha < 2.53 \times 10^{27}(1+\log(6n))^2.$$

With the help of Maple we find that n satisfies

$$n < 3 \times 10^{31}.$$

□

3.2. **The reduction procedure.** We note that the bound from Lemma 3.1 is too large for computational purposes. However, with the help of Lemmas 2.2 and 2.3, they can be considerably sharpened. This section is dedicated towards this goal.

Let

$$\Gamma_1 = s\log(\gamma) - n\log(\alpha) + \log(\sqrt{5}).$$

If $\Gamma_1 > 0$, using the fact that $x < \exp(x) - 1$ for all $x > 0$, together with the inequality (11) we obtain that

$$\Gamma_1 < \frac{5}{\alpha^{n-m}}.$$

If $\Gamma_1 < 0$, we have

$$|\Gamma_1| < \exp(-\Gamma_1) - 1 = \exp(-\Gamma_1)|\exp(\Gamma_1) - 1|.$$

Notice that for $n - m \geq 4$ we have

$$\exp(\Gamma_1) > 0.27,$$

which gives

$$\exp(-\Gamma_1) < 3.8.$$

Then by (11) we obtain

$$|\Gamma_1| < \frac{19}{\alpha^{n-m}}.$$

Therefore we have the inequality

$$\left| s\frac{\log(\gamma)}{\log(\alpha)} - n + \frac{\log(\sqrt{5})}{\log(\alpha)} \right| < 39.5\alpha^{-(n-m)}. \tag{14}$$

Let us now apply Lemma 2.2. For this we put

$$M = 6.1 \times 10^{31}, \quad w = n - m, \quad A = 39.5, \quad B = \alpha,$$

$$\psi = \frac{\log(\gamma)}{\log(\alpha)}, \quad \mu = \frac{\log(\sqrt{5})}{\log(\alpha)},$$

$$q_{60} = 722546402692124058304485813141551.$$

Note that

$$\varepsilon := \| \mu q_{60} \| - M \| \psi q_{60} \| > 0.153.$$

Since $s < M$ by (10) and Lemma 3.1, the inequality (14) has no solutions for

$$n - m \geq \frac{\log(Aq/\varepsilon)}{\log B} \geq 168.75.$$

This means that $n - m \leq 168$.

To reduce the bound on the integer n in Lemma 3.1 we apply again Lemma 2.2. We put

$$\Gamma_2 = s\log(\gamma) - n\log(\alpha) + \log\left(\frac{\sqrt{5}}{1 \pm \alpha^{m-n}}\right).$$

If $\Gamma_2 > 0$, by the inequality (12) we have

$$\Gamma_2 < \frac{21}{\alpha^n}.$$

If $\Gamma_2 < 0$, we have

$$|\Gamma_2| < \exp(-\Gamma_2) - 1 = \exp(-\Gamma_2)|\exp(\Gamma_2) - 1|.$$

Notice that

$$-\frac{21}{\alpha^n} < \exp(\Gamma_2) - 1 < \frac{21}{\alpha^n},$$

and for $n \geq 7$ we have

$$0.2 < 1 - \frac{21}{\alpha^n} < \exp(\Gamma_2).$$

Hence, we obtain $\exp(-\Gamma_2) < 5$ and it follows that

$$|\Gamma_2| < \frac{105}{\alpha^n}.$$

Summarizing we have the following inequality

$$\left| s\frac{\log(\gamma)}{\log(\alpha)} - n + \frac{\log\left(\frac{\sqrt{5}}{1\pm\alpha^{m-n}}\right)}{\log(\alpha)} \right| < \frac{218.2}{\alpha^n}. \tag{15}$$

We consider the following data:

$$M = 6.1\times 10^{31}, \quad w = n, \quad A = 218.2, \quad B = \alpha,$$

$$\psi = \frac{\log(\gamma)}{\log(\alpha)}, \quad \mu_{\pm,k} = \frac{\log\left(\frac{\sqrt{5}}{1\pm\alpha^{-k}}\right)}{\log(\alpha)}, \; k = 1, 2, \ldots, 178,$$

$$q_{66} = 276210093001120272437241265542247559.$$

A simple calculation with Maple shows that for $k = n - m \leq 168$,

$$\varepsilon \geq \| \mu_{\pm,94}\, q_{66} \| - M \| \psi q_{66} \| > 0.000499$$

unless $k = 2$ for $\mu_{+,k}$ or $k = 4$ for $\mu_{-,k}$ which are treated separately using Lemma 2.3 in the next paragraph. For $\varepsilon > 0.000499$ it follows from Lemma 2.2 that equation (1) can have solutions only for integers $n \leq 196$. By inequality (10) it follows that $s \leq 396$.

For the cases $k = 2$ for μ_+ or $k = 4$ for μ_- we make use of Lemma 2.3. Notice that for $k = 4$ we have $F_{n+4} - F_n = F_{n+3} + F_{n+2} - F_n = F_{n+3} + F_{n+1}$. Hence it remains to consider only the case $k = n - m = 2$ for which we have $\mu_{+,2} = \log\left(\frac{\sqrt{5}}{1+\alpha^{m-n}}\right) = \log(\alpha)$. Thus, inequality (15) becomes

$$\left| s\frac{\log(\gamma)}{\log(\alpha)} - n + 1 \right| < \frac{218.2}{\alpha^n}.$$

It follows that

$$\left| \frac{\log(\gamma)}{\log(\alpha)} - \frac{n-1}{s} \right| < \frac{218.2}{s\alpha^n}.$$

Assume that $n > 170$, since $s < M = 6.1\times 10^{31}$ we obtain that $\frac{218.2}{s\alpha^n} < \frac{1}{2s^2}$. By Lemma 2.3 the quantity $\frac{n-1}{s}$ is a convergent to $\frac{\log(\gamma)}{\log(\alpha)}$ whose continued fraction expansion is given by

$$\frac{\log(\gamma)}{\log(\alpha)} = [a_0, a_1, a_3, \ldots] = [0, 1, 1, 2, 2, 6, 2, 1, 2, 1, 2, 1, 1, 11, 1, 2, 3, 1, 7, 37, \ldots]$$

such that $q_{57} = 78018265498682576556029134678639$ is the first denominator of a convergent $\frac{p}{q}$ satisfying $q > M$. We obtain that

$$\frac{1}{(a(M)+2)s^2} \leq \left| \frac{\log(\gamma)}{\log(\alpha)} - \frac{n-1}{s} \right| < \frac{218.2}{s\alpha^n}, \text{ for any } s < M.$$

Since $a(M) := max\{a_i : i = 0, 1, 2, \ldots, 57\} = 64$ we have

$$\frac{1}{66s^2} < \frac{218.2}{s\alpha^n},$$

which gives that

$$n < \frac{\log\left(218.2(a(M)+2)M\right)}{\log(\alpha)} \leq 171.99.$$

Hence in both cases we have $n \leq 196$.

We use Maple to solve the Diophantine equations (1) in the ranges $1 \leq m \leq n \leq 300$ and $s \leq 604$, and we obtain only the solutions displayed in the statement of Theorems 1.1 and 1.2. This completes the proof of both theorems.

4. Perrin numbers as a product of two Fibonacci numbers

The aim of this section is to prove Theorem 1.3. We first show that solutions (n, m, s) of equation (2) satisfy $m \leq n < 194$ and $s < 776$, then we use a computer program to check such solutions.

We start with bounding s in terms of n. We make use again of the properties of Fibonacci and Perrin numbers to obtain

$$s < 4n. \tag{16}$$

4.1. The initial bound on n.

Now we reword equation (2) as

$$\frac{1}{5}(\alpha^n - \beta^n)(\alpha^m - \beta^m) = \gamma^s + \eta^s + \rho^s.$$

Using the fact that $\beta = -\alpha^{-1}$, we obtain

$$1 - 5\gamma^s\alpha^{-n-m} = \frac{5}{\alpha^{n+m}}(\eta^s + \rho^s) - (-\alpha^{-2})^{n+m} + (-\alpha^{-2})^n + (-\alpha^{-2})^m.$$

We set $\Lambda_3 = 5\gamma^s\alpha^{-n-m} - 1$. Then we have

$$|\Lambda_3| < \frac{13}{\alpha^m}. \tag{17}$$

Let us rewrite equation (2) as

$$\frac{\alpha^n - \beta^n}{\sqrt{5}}F_m = \gamma^s + \eta^s + \rho^s.$$

Reordering and taking the absolute value we obtain the following

$$\left|1 - \gamma^s\alpha^{-n}\sqrt{5}F_m^{-1}\right| \leq \frac{\sqrt{5}}{\alpha^n}\left(\frac{2}{F_m}|\eta|^s + \frac{1}{\sqrt{5}}\right).$$

This gives that

$$|\Lambda_4| < \frac{6}{\alpha^n}, \tag{18}$$

where $\Lambda_4 := \gamma^s\alpha^{-n}\frac{\sqrt{5}}{F_m} - 1$.

Lemma 4.1. *If $F_nF_m = E_s$ and $n \geq m$, then we have $n < 1.36 \times 10^{32}$.*

Proof. We apply Theorem 2.1 to $\Lambda_3 := 5\gamma^s\alpha^{-n-m} - 1$. We first note that, in a similar way as in the previous section, we can show that $\Lambda_3 \neq 0$. We take

$$l = 3, \quad \delta_1 = 5, \quad \delta_2 = \gamma, \quad \delta_3 = \alpha, \quad b_1 = 1, \quad b_2 = s \quad \text{and} \quad b_3 = -n-m.$$

We are still using the field $\mathbb{L} = \mathbb{Q}(\sqrt{5}, \gamma)$ which is of degree $D = 6$ over $\mathbb{Q}$. We have furthermore

$$h(\delta_1) = \log(5), \quad h(\delta_2) = \frac{\log \gamma}{3}, \quad \text{and} \quad h(\delta_3) = \frac{\log \alpha}{2}.$$

We now choose

$$A_1 = 10, \quad A_2 = 0.6, \quad A_3 = 1.5, \quad \text{and} \quad B = 4n.$$

With the above data, the theorem of Matveev gives that

$$|\Lambda_3| \geq \exp\left(-1.3 \times 10^{14} \times (1 + \log(4n))\right).$$

Using inequality (17) we obtain

$$m \log(\alpha) < 1.4 \times 10^{14}(1 + \log(4n)). \tag{19}$$

To be able to apply Matveev's result to Λ_4, we consider the necessary data, namely

$$l = 3, \quad \delta_1 = \sqrt{5}/F_m, \quad \delta_2 = \gamma, \quad \delta_3 = \alpha, \quad b_1 = 1, \quad b_2 = s \quad \text{and} \quad b_3 = -n.$$

Notice that $\delta_1 = \sqrt{5}/F_m = \dfrac{5}{\alpha^m - \beta^m}$, hence

$$h(\delta_1) \leq 2mh(\alpha) + \log(10) = m \log \alpha + \log(10).$$

Moreover, by $\alpha^{m-2} \leq F_m \leq \alpha^{m-1}$ we have

$$|\log \delta_1| < 6h(\delta_1).$$

Thus, by (19) we can take $A_1 = 8.41 \times 10^{14}(1 + \log(4n))$.

Now we can apply Matveev's theorem to Λ_4. More precisely, we obtain

$$|\Lambda_4| \geq \exp\left(-1.1 \times 10^{28}(1 + \log(4n))^2\right).$$

Hence, using (18) we obtain

$$n \log(\alpha) < 1.12 \times 10^{28}(1 + \log(4n))^2.$$

With the help of Maple we get that

$$n < 1.36 \times 10^{32}.$$

□

4.2. **The reduction procedure.** Now we need to reduce the bound obtained in Lemma 4.1. To do so, we use Lemma 2.2. Using the same methods as in the previous section on $\log(\Lambda_3 + 1)$ and $\log(\Lambda_4 + 1)$ we obtain for $m \geq 6$ the following inequalities:

$$\left| s\frac{\log(\gamma)}{\log(\alpha)} - (n + m) + \frac{5}{\log(\alpha)} \right| < 100.2\, \alpha^{-m}, \tag{20}$$

and

$$\left| s\frac{\log(\gamma)}{\log(\alpha)} - n + \frac{\log(\frac{\sqrt{5}}{F_m})}{\log(\alpha)} \right| < 104\, \alpha^{-n}. \tag{21}$$

For (20), we consider the following data:

$$\begin{aligned} M &= 5.44 \times 10^{32}, \quad w = m, \quad A = 100.2, \quad B = \alpha, \\ \psi &= \frac{\log(\gamma)}{\log(\alpha)}, \quad \mu = \frac{\log(5)}{\log(\alpha)}, \\ q_{63} &= 114786477749345061824556991553794 17. \end{aligned}$$

Notice that, by Lemma 4.1 and (16) we can see that $s < M$. Since $\varepsilon > 0.18$, the inequality (20) has no solutions for $m > 176.071$. Then the possible solutions exist only for $m \leq 176$.

Now we focus on (21). We have the following data:

$$\begin{aligned} M &= 5.44 \times 10^{32}, \quad w = n, \quad A = 104, \quad B = \alpha \\ \psi &= \frac{\log(\gamma)}{\log(\alpha)}, \quad \mu = \frac{\log\left(\frac{\sqrt{5}}{F_m}\right)}{\log(\alpha)}, \\ q_{66} &= 2762100930011202724372412655422475 59. \end{aligned}$$

With the help of Maple, this data and Lemma 2.2 give that the inequality (21) has no solutions for $n > 194.595$. Then, it has possible solutions only for $n \leq 194$.

We use Maple to solve the Diophantine equation (2) in the ranges $1 \leq m \leq n \leq 194$ and $s \leq 776$, and we obtain only the solutions displayed in the statement of Theorem 1.3. This completes the proof of the theorem.

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